A CONGRUENCE MODULO n^3 INVOLVING TWO CONSECUTIVE SUMS OF POWERS AND ITS APPLICATIONS

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Abstract. For various positive integers k, the sums of kth powers of the first n positive integers,

$$S_k(n+1) = 1^k + 2^k + \cdots + n^k,$$

have got to be some of the most popular sums in all of mathematics. In this note we prove that for each $k \geq 2$

$$2S_{2k+1}(n) - (2k+1)nS_{2k}(n) \equiv \begin{cases} 0 \pmod{n^3} & \text{if } k \text{ is even or } n \text{ is odd} \\ & \text{or } n \equiv 0 \pmod 4 \\ \frac{n^3}{2} \pmod{n^3} & \text{if } k \text{ is odd} \\ & \text{ and } n \equiv 2 \pmod 4). \end{cases}$$

The above congruence allows us to state an equivalent formulation of Giuga's conjecture. Moreover, we prove that the first above congruence is satisfied modulo n^4 whenever $n \geq 5$ is a prime number such that $n-1 \nmid 2k-2$. In particular, this congruence arises a conjecture for a prime to be Wolstenholme prime. We also propose several Giuga-Agoh's-like conjectures. Further, we establish two congruences modulo n^3 for two binomial type sums involving sums of powers $S_{2i}(n)$ with $i = 0, 1, \ldots, k$. Furthermore, using the above congruence reduced modulo n^2, we obtain an extension of Carlitz-von Staudt result for odd power sums.

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1. Introduction and Basic Result

The sum of powers of integers $\sum_{i=1}^{n} i^k$ is a well-studied problem in mathematics (see e.g., [9], [37]). Finding formulas for these sums has interested mathematicians for more than 300 years since the time of James Bernoulli (1665-1705). Several methods were used to find the sum $S_k(n)$ (see for example Vakil [46]). These lead to numerous recurrence relations. For a nice account of sums of powers see [14]. For simplicity, here as always in the sequel, for all integers $k \geq 1$ and $n \geq 2$ we denote

$$S_k(n) := \sum_{i=1}^{n-1} i^k = 1^k + 2^k + 3^k + \cdots + (n-1)^k.$$

The study of these sums led Jakob Bernoulli to develop numbers later named in his honor. Namely, the celebrated *Bernoulli's formula* (sometimes called *Faulhaber's*

formula) gives the sum $S_k(n)$ explicitly as (see e.g., Beardon [4])

$$S_k(n)=\frac{1}{k+1}\sum_{i=0}^{k}\binom{k+1}{i}n^{k+1-i}B_i \tag{1.1}$$

where B_i $(i=0,1,2,\dots)$ are *Bernoulli numbers* defined by the generating function

$$\sum_{i=0}^{\infty}B_i\frac{x^i}{i!}=\frac{x}{e^x-1}.$$

It is easy to find the values $B_0=1$, $B_1=-\frac{1}{2}$, $B_2=\frac{1}{6}$, $B_4=-\frac{1}{30}$, and $B_i=0$ for odd $i\ge 3$. Furthermore, $(-1)^{i-1}B_{2i}>0$ for all $i\ge 1$. These and many other properties can be found, for instance, in [21]. Several generalizations of the formula (1.1) were established by Z.-H. Sun ([43, Theorem 2.1] and [44]) and Z.-W. Sun ([45]).

By the well known *Pascal's identity* proven by Pascal in 1654 (see e.g., [27])

$$\sum_{i=0}^{k-1}\binom{k}{i}S_i(n+1)=(n+1)^k-1. \tag{1.2}$$

Recall also that the formula (1.2) is also presented in Bernoulli's *Ars Conjectandi* [6], (also see [17, pp. 269–270]) published posthumously in 1713.

On the other hand, divisibility properties of the sums $S_k(n)$ were investigated by many authors (see [13], [25], [28], [39]). For example, in 2003 P. Damianou and P. Schumer [13, Theorem, p. 219] proved:

1) if k is odd, then n divides $S_k(n)$ if and only if n is incogruent to 2 modulo 4;

2) if k is even, then n divides $S_k(n)$ if and only if n is not divisible by any prime p such that $p\mid D_k$, where D_k is the denominator of the kth Bernoulli number B_k.

Motivated by the recurrence formula for $S_k(n)$ recently obtained by the author in [31, Corollary 1.7], in this note we prove the following basic result.

Theorem 1.1. *Let k and n be positive integers. Then for each $k\ge 2$*

$$2S_{2k+1}(n)-(2k+1)nS_{2k}(n)\equiv\begin{cases}0 \pmod{n^3} & \text{if } k \text{ is even or } n \text{ is odd}\\ & \text{or } n\equiv 0 \pmod 4\\ \frac{n^3}{2}\pmod{n^3} & \text{if } k \text{ is odd}\\ & \text{and } n\equiv 2 \pmod 4.\end{cases} \tag{1.3}$$

Furthermore,

$$2S_3(n)-3nS_2(n)\equiv\begin{cases}0 \pmod{n^3} & \text{if } n \text{ is odd}\\ \frac{n^3}{2}\pmod{n^3} & \text{if } n \text{ is even.}\end{cases} \tag{1.4}$$

In particular, for all $k\ge 1$ and $n\ge 1$, we have

$$2S_{2k+1}(n)\equiv(2k+1)nS_{2k}(n)\pmod{n^2}, \tag{1.5}$$

and for all $k\ge 1$ and $n\not\equiv 2 \pmod 4$

$$2S_{2k+1}(n)\equiv(2k+1)nS_{2k}(n)\pmod{n^3}. \tag{1.6}$$

Combining the congruence (1.5) and the "even case" 2) of a result of Damianou and Schumer [13, Theorem, p. 219] mentioned above, we obtain the following "odd" extension of their result.

Corollary 1.2. *If k is an odd positive integer and n a positive integer such that n is not divisible by any prime p such that $p \mid D_{k-1}$, where D_{k-1} is the denominator of the $(k-1)$th Bernoulli number B_{k-1}. Then n^2 divides $2S_k(n)$.*

Conversely, if k is an odd positive integer and n a positive integer relatively prime to k such that n^2 divides $2S_k(n)$, then n is not divisible by any prime p such that $p \mid D_{k-1}$, where D_{k-1} is the denominator of the $(k-1)$th Bernoulli number B_{k-1}.

The paper is organized as follows. Some applications of Theorem 1.1 are presented in the following section. In Subsection 2.1 we give three particular cases of the congruence (1.3) of Theorem 1.1 (Corollary 2.1). One of these congruences immediately yields a reformulation of Giuga's conjecture in terms of the divisibility of $2S_n(n)+n^2$ by n^3 (Proposition 2.2).

In the next subsection we establish the fact that the congruence (1.6) is satisfied modulo n^4 whenever $n \geq 5$ is a prime number such that $n-1 \nmid 2k-2$ ((2.7) of Proposition 2.5). Motivated by some particular cases of this congruence and related computations via `Mathematica 8`, we propose several Giuga-Agoh's-like conjectures. In particular, Conjecture 2.10 characterizes Wolstenholme primes as positive integers n such that $S_{n-2}(n) \equiv 0 (\bmod n^3)$.

In Subsection 2.3 we establish two congruences modulo n^3 for two binomial sums involving sums of powers $S_{2i}(n)$ with $i = 0, 1, \dots, k$ (Proposition 2.15).

Combining the congruence (1.5) of Theorem 1.1 with Carlitz-von Staudt result for determining $S_{2k}(n) (\bmod n)$ (Theorem 2.18), in the last subsection of Section 2 we extend this result modulo n^2 for power sums $S_{2k+1}(n)$ (Theorem 2.21). We believe that Theorem 2.21 can be useful for study some Erdős-Moser-like Diophantine equations with odd k.

Proofs of all our results are given in Section 3.

2. Applications of Theorem 1.1

2.1. **Variations of Giuga-Agoh's conjecture.** Taking $k = (n-1)/2$ if n is odd, $k = n/2$ if n is even and $k = (n-2)/2$ if n is even into (1.3) of Theorem 1.1, we respectively obtain the following three congruences.

Corollary 2.1. *If n is an odd positive integer, then*

$$2S_n(n) \equiv n^2 S_{n-1}(n) \pmod{n^3}. \tag{2.1}$$

If n is even, then

$$2S_{n+1}(n) - n(n+1)S_n(n) \equiv \begin{cases} 0 \pmod{n^3} & if\ n \equiv 0 (\bmod 4) \\ \frac{n^3}{2} \pmod{n^3} & if\ n \equiv 2 (\bmod 4) \end{cases} \tag{2.2}$$

and

$$2S_{n-1}(n) \equiv n(n-1)S_{n-2}(n) \pmod{n^3}. \tag{2.3}$$

In particular, for each even n we have

$$S_{n-1}(n) \equiv \begin{cases} 0 \pmod{n} & if\ n \equiv 0 \pmod 4 \\ \frac{n}{2} \pmod{n} & if\ n \equiv 2 \pmod 4). \end{cases} \tag{2.4}$$

Notice that if n is any prime, then by Fermat's little theorem, $S_{n-1}(n) \equiv -1(\bmod\ n)$. In 1950 G. Giuga [16] conjectured that a positive integer $n \geq 2$ is a prime if and only if $S_{n-1}(n) \equiv -1(\bmod\, n)$. The following proposition provides an equivalent formulation of Giuga's conjecture.

Proposition 2.2. *The following conjectures are equivalent:*

(i) *A positive integer $n \geq 3$ is a prime if and only if*

$$S_{n-1}(n) \equiv -1 \pmod{n}. \tag{2.5}$$

(ii) *A positive integer $n \geq 3$ is a prime if and only if*

$$2S_n(n) \equiv -n^2 \pmod{n^3}. \tag{2.6}$$

Since by congruence (2.4), $S_{n-1}(n) \not\equiv -1(\bmod\, n)$ for each even $n \geq 4$, without loss of generality Giuga's conjecture may be restricted to the set of odd positive integers. In view of this fact and the fact that by (2.1), $n^2 \mid S_n(n)$ for each odd n, Proposition 2.2 yields the following equivalent formulation of Giuga's conjecture.

Conjecture 2.3 (Giuga's conjecture)**.** *An odd integer $n \geq 3$ is is a prime if and only if*

$$\frac{2S_n(n)}{n^2} \equiv -1 \pmod{n}.$$

Remark 2.4. It is known that $S_{n-1}(n) \equiv -1 \,(\bmod\, n)$ if and only if for each prime divisor p of n, $(p-1) \mid (n/p-1)$ and $p \mid (n/p-1)$ (see [16], [7, Theorem 1]). Therefore, any counterexample to Giuga's conjecture must be squarefree. Giuga [16] showed that there are no exceptions to the conjecture up to 10^{1000}. In 1985 Bedocchi [5] improved this bound to $n > 10^{1700}$. Finally, in 1996 D. Borwein, J.M. Borwein, P.B. Borwein and R. Girgensohn raised the bound to $n > 10^{13887}$. Recently, F. Luca, C. Pomerance and I. Shparlinski [26] proved that for any real number x, the number of counterexamples to Giuga's conjecture $G(x) := \#\{n < x : n \text{ is composite and } S_{n-1}(n) \equiv -1 \,(\bmod\, n)\}$ satisfies the estimate $G(x) = o(\sqrt{x})$ as $x \to \infty$.

Independently, in 1990 T. Agoh (published in 1995 [1]; see also [8] and Sloane's sequence A046094 in [38]) conjectured that a positive integer $n \geq 2$ is a prime if and only if $nB_{n-1} \equiv -1(\bmod\, n)$. Note that the denominator of the number nB_{n-1} can be greater than 1, but since by von Staudt-Clausen's theorem (1840) (see e.g., [20, Theorem 118]; cf. the equality (2.18) given below), the denominator of any Bernoulli number B_{2k} is squarefree, it follows that the denominator of nB_{n-1} is invertible modulo n. In 1996 it was reported by T. Agoh [7] that his conjecture is equivalent to Giuga's conjecture, hence the name Giuga-Agoh's conjecture found in the litterature. It was pointed out in [7] that this can be seen from the Bernoulli formula (1.1) after some analysis involving von Staudt-Clausen's theorem. The equivalence of both conjectures is in details proved in 2002 by B.C. Kellner [22, Satz 3.1.3, Section 3.1, p. 97] (also see [23, Theorem 2.3]).

Quite recently, J.M. Grau and A.M. Oller-Marcén [18, Corollary 4] proved that an integer n is a counterexample to Giuga's conjecture if and only if it is both a Carmichael and a Giuga number (for definitions and more information on Carmichael numbers see W.R. Alford et al. [2] and W.D. Banks and C. Pomerance [3], and for Giuga numbers see D. Borwein et al. [7], J.M. Borwein and E. Wong [8], and E. Wong [47, Chapter 2]; also see Sloane's sequences A007850 and A002997 [38]). Furthermore, several open problems concerning Giuga's conjecture can be found in [**?**, 8, E Open Problems]

2.2. **The congruence (1.3) holds modulo n^4 for a prime $n \geq 5$.** The following result shows that for each prime $n \geq 5$ the first congruence of (1.3) is also satisfied modulo n^4.

Proposition 2.5. *Let $p \geq 5$ be a prime and let $k \geq 2$ be a positive integer such that $p-1 \nmid 2k-2$. Then*

$$2S_{2k+1}(p) \equiv (2k+1)pS_{2k}(p) \pmod{p^4}. \tag{2.7}$$

Furthermore, if $p-1 \nmid 2k$, then

$$S_{2k-1}(p) \equiv 0 \pmod{p^2}. \tag{2.8}$$

As a consequence of Proposition 2.5, we obtain the following "supercongruence" which generalizes Lemma 2.4 in [30].

Corollary 2.6. *Let $p \geq 5$ be a prime and let k be a positive integer such that $k \leq (p^4-p^3-4)/2$ and $p-1 \nmid 2k+2$. Then*

$$2R_{2k-1}(p) \equiv (1-2k)pR_{2k}(p) \pmod{p^4} \tag{2.9}$$

where

$$R_s(p) := \sum_{i=1}^{p-1} \frac{1}{i^s}, \quad s = 1, 2, \ldots .$$

Remark 2.7. Z.-H. Sun [42, Section 5, Theorem (5.1)] in terms of Bernoulli numbers explicitly determined $\sum_{i=1}^{p-1}(1/i^k) \pmod{p^3}$ for each prime $p \geq 5$ and $k = 1, 2, \ldots, p-1$. In particular, substituting the second congruence of Theorem 5.1(a) in [42] (with $2k$ instead of even k) into (2.7), we immediately obtain the following "supercongruence"

$$R_{2k-1}(p) \equiv \frac{k(1-2k)}{2}\left(\frac{B_{2p-2-2k}}{p-1-k} - 4\frac{B_{p-1-2k}}{p-1-2k}\right)p^2 \pmod{p^4}$$

for all primes $p \geq 7$ and $k = 1, \ldots, (p-5)/2$.

By [42, (5.1) on page 206],

$$S_{2k}(p) \equiv \frac{p}{3}(3B_{2k} + k(2k-1)p^2B_{2k-2}) \pmod{p^3}, \tag{2.10}$$

which inserting into (2.7) gives

$$S_{2k+1}(p) \equiv \frac{2k+1}{2}p^2B_{2k} \pmod{p^4} \tag{2.11}$$

for all primes $p \geq 5$ and positive integers $k \geq 2$ such that $p-1 \nmid 2k-2$. Moreover, (2.7) with $2k = p-1 \geq 4$ (i.e., $p \geq 5$) directly gives

$$S_p(p) \equiv \frac{p^2}{2} S_{p-1}(p) \pmod{p^4}.$$

Taking $2k+1 = p$ into (2.11), we find that

$$S_p(p) \equiv \frac{p^3}{2} B_{p-1} \pmod{p^4},$$

which reducing modulo p^3, and using the congruence $pB_{p-1} \equiv -1 (\bmod\, p)$, yields $2S_p(p) \equiv -p^2 (\bmod\, p^3)$. This is actually the congruence (2.6) of Proposition 2.2 with a prime $n = p \geq 5$.

Comparing the above two congruences gives $S_{p-1}(p) \equiv pB_{p-1} (\bmod\, p^2)$ for each prime $p \geq 5$. However, the congruence (2.10) with $2k = p-1$ implies that for all primes $p \geq 5$

$$S_{p-1}(p) \equiv pB_{p-1} \pmod{p^3}.$$

Remark 2.8. A computation shows that each of the congruences

$$S_n(n) \equiv \frac{n^3}{2} B_{n-1} \pmod{n^4}$$

and

$$S_{n-1}(n) \equiv nB_{n-1} \pmod{n^3}$$

is also satisfied for numerous odd composite positive integers n. However, we propose the following

Conjecture 2.9. *Each of the congruences*

$$S_n(n) \equiv \frac{n^3}{2} B_{n-1} \pmod{n^5},$$

$$S_{n-1}(n) \equiv nB_{n-1} \pmod{n^4}$$

is satisfied for none integer $n \geq 2$.

Similarly, taking $k = (p-3)/2$ into (2.11) for each prime $p \geq 5$ we get

$$S_{p-2}(p) \equiv \frac{(p-2)p^2}{2} B_{p-3} \pmod{p^4}. \tag{2.12}$$

Therefore, $p^3 \mid S_{p-2}(p)$ if and only if the numerator of the Bernoulli number B_{p-3} is divisible by p, and such a prime is said to be *Wolstenholme prime*. The only two known such primes are 16843 and 2124679, and by a result of McIntosh and Roettger from [29], these primes are the only two Wolstenholme primes less than 10^9. In view of the above congruence, and our computation via `Mathematica 8` up to $n = 20000$ we have the following two conjectures.

Conjecture 2.10. *A positive integer $n \geq 2$ is a Wolstenholme prime if and only if*

$$S_{n-2}(n) \equiv 0 \pmod{n^3}.$$

Conjecture 2.11. *The congruence*

$$S_{n-2}(n) \equiv 0 \pmod{n^4}$$

is satisfied for none integer $n \geq 2$.

Remark 2.12. Quite recently, inspired by Giuga's conjecture, J.M. Grau, F. Luca and A.M. Oller-Marcén [19] studied the odd positive integers n satisfying the congruence

$$S_{(n-1)/2}(n) \equiv 0 \,(\mathrm{mod}\, n).$$

The authors observed [19, Section 2, Proposition 2.1] that this congruence is satisfied for each odd prime n and for each odd positive integer $n \equiv 3 \,(\mathrm{mod}\, 4)$. Notice that if $n = 4k+3$ with $k \geq 0$, the first part of the congruence (1.3) yields

$$2S_{(n-1)/2}(n) \equiv \frac{(n-1)n}{2} S_{(n-3)/2}(n) \pmod{n^3}$$

which is by the congruence (2.7) satisfied modulo n^4 for each prime $n \geq 7$ such that $n \equiv 3 \,(\mathrm{mod}\, 4)$. Multiplying the above congruence by 2 and reducing the modulus, immediately gives

$$4S_{(n-1)/2}(n) \equiv -nS_{(n-3)/2}(n) \pmod{n^2}.$$

The above congruence shows that $S_{(n-1)/2}(n) \equiv 0 (\mathrm{mod}\, n^2)$ for some $n \equiv 3 \,(\mathrm{mod}\, 4)$ if and only if $S_{(n-3)/2}(n) \equiv 0 (\mathrm{mod}\, n)$. Furthermore, reducing the congruence (2.11) with $k = (p-3)/4$ where $p \geq 7$ is a prime such that $p \equiv 3 \,(\mathrm{mod}\, 4)$ gives

$$S_{(p-1)/2}(p) \equiv -\frac{p^2}{4} B_{(p-3)/2} \pmod{p^3}, \tag{2.13}$$

whence it follows that for such a prime p, $S_{(p-1)/2}(p) \equiv 0 (\mathrm{mod}\, p^2)$.

On the other hand, if $n \equiv 1 \,(\mathrm{mod}\, 4)$, that is $n = 4k+1$ with $k \geq 1$, the first part of the congruence (1.3) yields

$$2S_{(n+1)/2}(n) \equiv \frac{(n+1)n}{2} S_{(n-1)/2}(n) \pmod{n^3}$$

which is by the congruence (2.7) satisfied modulo n^4 for each prime $n \equiv 1 \,(\mathrm{mod}\, 4)$. Multiplying the above congruence by 2 and reducing the modulus, immediately gives

$$4S_{(n+1)/2}(n) \equiv nS_{(n-1)/2}(n) \pmod{n^2}.$$

The above congruence shows that $S_{(n-1)/2}(n) \equiv 0 (\mathrm{mod}\, n)$ for some $n \equiv 1 \,(\mathrm{mod}\, 4)$ if and only if $S_{(n+1)/2}(n) \equiv 0 (\mathrm{mod}\, n^2)$. For example, by [19, Proposition 2.3] (cf. Corollary 2.22 given below) both previous congruences are satisfied for every odd prime power $n = p^{2s+1}$ with any prime $p \equiv 1 \,(\mathrm{mod}\, 4)$ and a positive integer s. Moreover, reducing the congruence (2.10) with $k = (p-1)/4$ where $p \geq 5$ is a prime such that $p \equiv 1 \,(\mathrm{mod}\, 4)$ gives

$$S_{(p-1)/2}(p) \equiv pB_{(p-1)/2} \pmod{p^2}. \tag{2.14}$$

The congruence (2.14) shows that $S_{(p-1)/2}(p) \equiv 0 (\mathrm{mod}\, p^2)$ whenever $p \equiv 1 \,(\mathrm{mod}\, 4)$ is an irregular prime for which $B_{(p-1)/2} \equiv 0 (\mathrm{mod}\, p)$. That the converse is not true

shows the composite number $n = 3737 = 37 \times 101$ satisfying $S_{(n-1)/2}(n) \equiv 0(\bmod n^2)$ (this is the only such a composite number less than 16000).

Nevertheless, in view of the congruences (2.13) and using similar arguments preceeding Conjecture 2.10 (including a computation up to $n = 20000$), we have the following conjecture.

Conjecture 2.13. *An odd positive integer $n \geq 3$ such that $n \equiv 3 \pmod 4$ satisfies the congruence*

$$S_{(n-1)/2}(n) \equiv 0 \pmod{n^3}$$

if and only if n is an irregular prime for which $B_{(n-3)/2} \equiv 0(\bmod n)$.

We also propose the following

Conjecture 2.14. *The congruence*

$$S_{(n-1)/2}(n) \equiv 0 \pmod{n^3}$$

is satisfied for none odd positive integer $n \geq 5$ such that $n \equiv 1 \pmod 4$.

2.3. **Two congruences modulo n^3 involving power sums $S_k(n)$.** Combining the congruences of Theorem 1.1 with Pascal's identity, we can arrive to the following congruences.

Proposition 2.15. *Let k and n be positive integers. Then*

$$2\sum_{i=0}^{k}(1+n(k+1-i))\binom{2k+2}{2i}S_{2i}(n) \equiv -2 \pmod{n^3} \tag{2.15}$$

and

$$2\sum_{i=0}^{k}\left(\binom{2k+2}{2i}+n(k+1)\binom{2k+1}{2i}\right)S_{2i}(n) \equiv -2 \pmod{n^3}. \tag{2.16}$$

Remark 2.16. Clearly, if n is odd, then both congruences (2.15) and (2.16) remain also valid after dividing by 2. However, a computation via `Mathematica 8` for small values k and even n suggests that this would be true for each k and even n, and so we have

Conjecture 2.17. *The congruence*

$$\sum_{i=0}^{k}(1+n(k+1-i))\binom{2k+2}{2i}S_{2i}(n) \equiv -1 \pmod{n^3}$$

is satisfied for all $k \geq 1$ and even n.

2.4. **An extension of Carlitz-von Staudt result for odd power sums.** The following congruence is known as a *Carlitz-von Staudt's result* [10] in 1961 (for an easier proof see [34, Theorem 3]).

Theorem 2.18. ([10], [34, Theorem 3]) *Let k and $n>1$ be positive integers. Then*

$$S_k(n)\equiv\begin{cases}0\pmod{\frac{(n-1)n}{2}} & if\ k\ is\ odd\\ -\sum_{(p-1)|k,p|n}\frac{n}{p}\pmod n & if\ k\ is\ even\end{cases}\tag{2.17}$$

where the summation is taken over all primes p such that $(p-1)\mid k$ and $p\mid n$.

Remark 2.19. It is easy to show the first ("odd") part of Theorem 2.18 (see e.g., [34, Proof of Theorem 3] or [28, Proposition 1]) whose proof is a modification of Lengyel's arguments in [25]. Recall also that the classical *theorem of Faulhaber* states that every sum $S_{2k-1}(n)$ (of odd power) can be expressed as a polynomial of the triangular number $T_{n-1}:=(n-1)n/2$; see e.g., [4] or [24]. For even powers, it has been shown that the sum $S_{2k}(n)$ is a polynomial in the triangular number T_{n-1} multiplied by a linear factor in n (see e.g., [24]).

Remark 2.20. The second part of the congruence (2.17) in Theorem 2.18 can be proved using the famous von Staudt-Clausen theorem (given below) discovered without proof by C. Clausen [12] in 1840, and independently by K.G.C. von Staudt in 1840 [40]; for alternative proofs, see, e.g., Carlitz [10], Moree [32] or Moree [34, Theorem 3]. This also follows from Chowla's proof of von Staudt-Clausen theorem [11]. Namely, Chowla proved that the difference

$$\frac{S_{2k}(n+1)}{n}-B_{2k}$$

is an integer for all positive integers k and n. This together with the facts that $S_{2k}(n+1)\equiv S_{2k}(n)\pmod n$ and that by *von Staudt-Clausen theorem*,

$$B_{2k}=A_{2k}-\sum_{\substack{(p-1)|2k\\ p\ \mathrm{prime}}}\frac{1}{p},\tag{2.18}$$

where A_{2k} is an integer, immediately gives the second part of the congruence (2.17).

Recall also that in many places, the von Staudt-Clausen theorem is stated in the following equivalent statement (e.g., see [41, page 153]):

$$pB_{2k}\equiv\begin{cases}0\pmod p & \text{if } p-1\nmid 2k\\ -1\pmod p & \text{if } p-1\mid 2k,\end{cases}$$

where p is a prime and k a positive integer.

Combining the congruence (1.5) in Theorem 1.1 with the second ("even") part of the congruence (2.17), we immediately obtain an improvement of its first ("odd") part as follows.

Theorem 2.21. *Let Let k and n be positive integers. Then*

$$2S_{2k+1}(n)\equiv-(2k+1)n\sum_{(p-1)|2k,p|n}\frac{n}{p}\pmod{n^2},\tag{2.19}$$

where the summation is taken over all primes p such that $p-1\mid k$ and $p\mid n$.

In particular, taking $n = p^s$ and $k = (p^s - 1)/4$ into (2.19) where p is an odd prime p and $s \geq 1$ such that $p^s \equiv 1 (\bmod 4)$, we immediately obtain an analogue of Proposition 2.3 in a recent paper [19].

Corollary 2.22. *Let p be an odd prime. Then*

$$S_{(p^s+1)/2}(p^s) \equiv \begin{cases} 0 \, (\bmod \, p^{2s}) & if \ p \equiv 1 (\bmod 4) \ and \ s \geq 1 \ is \ odd \\ -\frac{p^{2s-1}}{4} \, (\bmod \, p^{2s}) & if \ s \geq 2 \ is \ even. \end{cases}$$

Finally, comparing (2.17), (2.18) and (2.19), we immediately obtain an "odd" extension of a result due to Kellner [23, Theorem 1.2] in 2004 (the congruence (2.20) given below).

Corollary 2.23. *Let Let k and n be positive integers. Then*

$$S_{2k}(n) \equiv nB_{2k} \pmod{n} \quad (Kellner \ [20]) \tag{2.20}$$

and

$$2S_{2k+1}(n) \equiv (2k+1)n^2 B_{2k} \pmod{n^2}. \tag{2.21}$$

Remark 2.24. Notice also that Theorem 2.18 plays a key role in a recent study ([32], [34], [15], [35]) of the *Erdős-Moser Diophantine equation*

$$1^k + 2^k + \cdots + (m-2)^k + (m-1)^k = m^k \tag{2.22}$$

where $m \geq 2$ and $k \geq 2$ are positive integers. Notice that $(m, k) = (3, 1)$ is the only solution for $k = 1$. In letter to Leo Moser around 1950, Paul Erdős conjectured that such solutions of the above equation do not exist (see [36]). Using remarkably elementary methods, Moser [36] showed in 1953 that if (m, k) is a solution of (2.22) with $m \geq 2$ and $k \geq 2$, then $m > 10^{10^6}$ and k is even. Recently, using Theorem 2.18, P. Moree [34, Theorem 4] improved the bound on m to $1.485 \cdot 10^{9321155}$. That Theorem 2.18 can be used to reprove Moser's result was first observed in 1996 by Moree [33], where it played a key role in the study of the more general equation

$$1^k + 2^k + \cdots + (m-2)^k + (m-1)^k = am^k \tag{2.23}$$

where a is a given positive integer. Moree [33] generalized Erdős-Moser conjecture in the sense that the only solution of the "generalized" Erdős-Moser Diophantine equation (2.23) is the trivial solution $1 + 2 + \cdots + 2a = a(2a + 1)$. Notice also that Moree [33, Proposition 9] proved that in any solution of the equation (2.23), m is odd. Nevertheless, motivated by the Moser's technique [34, proof of Theorem 3] previously mentioned, to study (2.22), we believe that Theorem 2.21 can be useful for study some other Erdős-Moser type Diophantine equations with odd k.

3. Proofs of Theorem 1.1, Corollaries 1.2, 2.6 and Propositions 2.2, 2.5 and 2.15

Proof of Theorem 1.1. If $k\ge 1$ then by the binomial formula, for each $i=1,2,\dots,n-1$ we have

$$\begin{aligned}&2(i^{2k+1}+(n-i)^{2k+1})-(2k+1)n(i^{2k}+(n-i)^{2k})\\ \equiv&2\left(i^{2k+1}-i^{2k+1})+\binom{2k+1}{1}ni^{2k}-\binom{2k+1}{2}n^2i^{2k-1}\right)\\ &-(2k+1)n\left(i^{2k}+i^{2k}-\binom{2k}{1}ni^{2k-1}\right)\pmod{n^3}\\ =&2(2k+1)ni^{2k}-2(2k+1)kn^2i^{2k-1}-2(2k+1)ni^{2k}+2(2k+1)kn^2i^{2k-1}\\ =&0\pmod{n^3}.\end{aligned}\tag{3.1}$$

If $k\ge 3$ and n is odd then after summation of (3.1) over $i=1,2,\dots,(n-1)/2$ we obtain

$$2\sum_{i=1}^{n-1}i^{2k+1}-(2k+1)n\sum_{i=1}^{n-1}i^{2k}\equiv 0\pmod{n^3}.\tag{3.2}$$

If $k\ge 2$ and n is even then after summation of (3.1) over $i=1,2,\dots,n/2$ we get

$$2\sum_{i=1}^{n-1}i^{2k+1}+2\left(\frac{n}{2}\right)^{2k+1}-(2k+1)n\sum_{i=1}^{n-1}i^{2k}-(2k+1)n\left(\frac{n}{2}\right)^{2k}\equiv 0\pmod{n^3},$$

or equivalently,

$$2S_{2k+1}-(2k+1)nS_{2k}\equiv\frac{kn^{2k+1}}{2^{2k-1}}=\frac{n^3}{2}\cdot k\cdot\left(\frac{n}{2}\right)^{2k-2}\pmod{n^3}.\tag{3.3}$$

Since for even n

$$\frac{n^3}{2}\cdot k\cdot\left(\frac{n}{2}\right)^{2k-2}\equiv\begin{cases}0\,(\mathrm{mod}\,n^3) & \text{if } k \text{ is even or } n\equiv 0\,(\mathrm{mod}\,4)\\ \frac{n^3}{2}\,(\mathrm{mod}\,n^3) & \text{if } k \text{ is odd and } n\equiv 2\,(\mathrm{mod}\,4),\end{cases}$$

this together with (3.3) and (3.2) yields both congruences of (1.3) in Theorem 1.1.

Finally, for $k=1$ we have

$$2S_3(n)-3nS_2(n)=\frac{n^3}{2}\cdot(1-n)\equiv\begin{cases}0\pmod{n^3} & \text{if } n \text{ is odd}\\ \frac{n^3}{2}\pmod{n^3} & \text{if } n \text{ is even.}\end{cases}$$

This completes the proof. □

Proof of Corollary 1.2. Both assertions immediately follow applying the congruence (1.5) and a result of P. Damianou and P. Schumer [13, Theorem, p. 219] which asserts that if k is even, then n divides $S_k(n)$ if and only if n is not divisible by any prime p such that $p\mid D_k$, where D_k is the denominator of the kth Bernoulli number B_k. □

Proof of Proposition 2.2. *Proof of* $(i) \Rightarrow (ii)$. Suppose that Giuga's conjecture is true. Then if n is an odd positive integer satisfying the congruence (2.6) of Proposition 2.2, using this and (2.1) of Corollary 2.1, we find that

$$n^2 S_{n-1}(n) \equiv 2S_n(n) \equiv -n^2 \pmod{n^3},$$

whence we have

$$S_{n-1}(n) \equiv -1 \pmod{n}.$$

By Giuga's conjecture, the above congruence implies that n is a prime.

If $n \geq 4$ is an even positive integer, then the congruence (2.4) shows that $S_{n-1}(n) \not\equiv -1 (\bmod\, n)$. We will show that for such a n, $2S_n(n) \not\equiv -n^2 (\bmod\, n^3)$. Take $n = 2^s(2l-1)$, where s and l are positive integers. Since for $i = 1, 2, \ldots$ we have $(2i)^n \equiv 0 (\bmod\, 2^n)$, this together with the inequality $2^{2^s} \geq 2^{s+1}$ yields $(2i)^n \equiv 0 (\bmod\, 2^{s+1})$. Therefore, we obtain

$$2S_n(n) \equiv 2 \sum_{\substack{1 \leq j \leq n-1 \\ j \text{ odd}}} j^n \pmod{2^{s+1}}.$$

Since by Euler theorem, for each odd j

$$j^n = j^{2^s(2l-1)} = \left(j^{2^s}\right)^{2l-1} = \left(j^{\varphi(2^{s+1})}\right)^{2l-1} \equiv 1 \pmod{2^{s+1}} \equiv 1 \pmod{2^s},$$

where $\varphi(m)$ is the Euler's totient function, then substituting this into above congruence, we get

$$2S_n(n) \equiv n = 2^s(2l-1) \not\equiv 0 \pmod{2^{s+1}}.$$

Now, if we suppose that $2S_n(n) \equiv -n^2 (\bmod\, n^3)$, then must be $2S_n(n) \equiv 0 (\bmod\, n^2)$, and so, $2S_n(n) \equiv 0 (\bmod\, 2^{2s}) \equiv 0 (\bmod\, 2^{s+1})$. This contradicts the above congruenece, and the impication $(i) \Rightarrow (ii)$ is proved.

Proof of $(ii) \Rightarrow (i)$. Now suppose that Conjecture (ii) of Proposition 2.2 is true. Then if n is an odd positive integer satisfying the congruence (2.5), multiplying this by n^2 and using (2.1) of Corollary 2.1, we find that

$$2S_n(n) \equiv n^2 S_{n-1}(n) \equiv -n^2 \pmod{n^3},$$

which implies that

$$2S_n(n) \equiv -n^2 \pmod{n^3}.$$

By our Conjecture (ii), the above congruence implies that n is a prime.

If $n \geq 4$ is an even positive integer, then we have previously shown that for such a n, $2S_n(n) \not\equiv -n^2 (\bmod\, n^3)$ and $S_{n-1}(n) \not\equiv -1 (\bmod\, n)$. This completes the proof of impication $(ii) \Rightarrow (i)$. □

Proof of Proposition 2.5. If we extend the congruence (3.1) modulo n^4, then in the same manner we obtain

$$\begin{aligned} &2(i^{2k+1} + (n-i)^{2k+1}) - (2k+1)n(i^{2k} + (n-i)^{2k}) \\ &\equiv 2\binom{2k+1}{3} n^3 i^{2k-2} - (2k+1)\binom{2k}{2} n^3 i^{2k-2} \pmod{n^4}, \end{aligned}$$

whence it follows that

$$2S_{2k+1}(n)-(2k+1)nS_{2k}(n)\equiv\frac{k(1-4k^2)}{3}n^3S_{2k-2}(n)\pmod{n^4}. \tag{3.4}$$

If $n=p$ is a prime such that $p-1\nmid 2k-2$, then the well known congruence $S_{2k-2}(p)\equiv 0\,(\bmod\, p)$ (see e.g., [43, the congruence (6.3)] or [27, Theorem 1]) and (3.4) yield the congruence (2.7). Finally, (2.8) immediately follows reducing (2.7) modulo p^2 and using the previous fact that $S_{2k}(p)\equiv 0\,(\bmod\, p)$ whenever $p-1\nmid 2k$. $\square$

Remark 3.1. Applying a result of P. Damianou and P. Schumer [13, Theorem, p. 219] used in the proof of Proposition 2.5 to the congruence (3.4), it follows that

$$2S_{2k+1}(n)\equiv(2k+1)nS_{2k}(n)\pmod{n^4}$$

whenever n is not divisible by any prime p such that $p\mid D_{2k-2}$, where D_{2k-2} is the denominator of the $(2k-2)$th Bernoulli number B_{2k-2}. The converse assertion is true if n is relatively prime to the integer $k(1-4k^2)/3$.

Proof of Corollary 2.6. By Euler's theorem [20], for all positive integers m and i such that $1\le m<p^4-p^3$ and $1\le i\le p-1$ we have $1/i^m\equiv i^{\varphi(p^4)-m}\pmod{p^4}$, where $\varphi(p^4)=p^4-p^3$ is the Euler's totient function. Therefore, $R_m\equiv S_{p^4-p^3-m}\,(\bmod\, p^4)$. Applying the last congruence for $m=2k-1$ and $m=2k$, and substituting this into (2.7) of Proposition 2.5 with $p^4-p^3-2k\ge 4$ instead of $2k$, we immediately obtain

$$2R_{2k-1}(p)\equiv(p^4-p^3-2k+1)pR_{2k}(p)\equiv(1-2k)pR_{2k}(p)\pmod{p^4},$$

as desired. $\square$

Proof of Proposition 2.15. As $S_0(n)=n-1$ and $S_1(n)=(n-1)n/2$, Pascal's identity (1.2) yields

$$\begin{aligned}2(n^{2k+2}-1)&=2\sum_{i=0}^{2k+1}\binom{2k+2}{i}S_i(n)\\&=2(n-1)(1+(k+1)n)+\sum_{i=1}^{k}\left(2\binom{2k+2}{2i}S_{2i}(n)+2\binom{2k+2}{2i+1}S_{2i+1}(n)\right).\end{aligned} \tag{3.5}$$

If n is odd, then multiplying the congruence (1.6) of Theorem 1.1 by $\binom{2k+2}{2i+1}$ and using the identity $\binom{2k+2}{2i+1}=\frac{2k+2-2i}{2i+1}\binom{2k+2}{2i}$, we find that

$$\begin{aligned}\binom{2k+2}{2i+1}2S_{2i+1}(n)&\equiv\frac{2k+2-2i}{2i+1}\binom{2k+2}{2i}(2i+1)nS_{2i}(n)\pmod{n^3}\\&=(2k+2-2i)\binom{2k+2}{2i}nS_{2i}(n)\pmod{n^3}\end{aligned} \tag{3.6}$$

for each $i=1,\dots,k$. Now substituting (3.6) into (3.5), we obtain

$$2(n-1)(1+(k+1)n)+2\sum_{i=1}^{k}(1+n(k+1-i))\binom{2k+2}{2i}S_{2i}(n)\equiv-2\pmod{n^3}, \tag{3.7}$$

which is obviously the same as (2.15).

If n is even, then since $\binom{2k+2}{2i+1}$ is even (this is true by the identity $\binom{2k+2}{2i+1} = \frac{2(k+1)}{2i+1}\binom{2k+1}{2i}$), we have that $\binom{2k+2}{2i+1}\frac{n^3}{2} \equiv 0 \,(\mathrm{mod}\, n^3)$. This shows that (2.15) is satisfied for even n and each $i = 1, \dots, k$, and hence, proceeding in the same manner as in the previous case, we obtain (2.15).

Further, applying the identities $2i\binom{2k+2}{2i} = (2k+2)\binom{2k+1}{2i-1}$ and $\binom{2k+2}{2i} - \binom{2k+1}{2i-1} = \binom{2k+1}{2i}$, the left hand side of (2.16) is equal to

$$\begin{aligned}
&2(1+n(k+1))\sum_{i=0}^{k}\binom{2k+2}{2i}S_{2i}(n) - n\sum_{i=0}^{k}2i\binom{2k+2}{2i}S_{2i}(n)\\
=&2\sum_{i=0}^{k}\binom{2k+2}{2i}S_{2i}(n) + 2n(k+1)(n-1) + 2n(k+1)\sum_{i=1}^{k}\binom{2k+2}{2i}S_{2i}(n)\\
&-2n(k+1)\sum_{i=1}^{k}\binom{2k+1}{2i-1}S_{2i}(n)\\
=&2\sum_{i=0}^{k}\binom{2k+2}{2i}S_{2i}(n) + 2n(k+1)(n-1)\\
&+2n(k+1)\sum_{i=1}^{k}\left(\binom{2k+2}{2i} - \binom{2k+1}{2i-1}\right)S_{2i}(n)\\
=&2\sum_{i=0}^{k}\binom{2k+2}{2i}S_{2i}(n) + 2n(k+1)(n-1) + 2n(k+1)\sum_{i=1}^{k}\binom{2k+1}{2i}S_{2i}(n)\\
=&2\sum_{i=0}^{k}\binom{2k+2}{2i}S_{2i}(n) + 2n(k+1)\sum_{i=0}^{k}\binom{2k+1}{2i}S_{2i}(n)\\
=&2\sum_{i=0}^{k}\left(\binom{2k+2}{2i} + n(k+1)\binom{2k+1}{2i}\right)S_{2i}(n).
\end{aligned}$$

Comparing the above equality with (2.15) immediately gives (2.16). □

References

[1] T. Agoh, 'On Giuga's conjecture', *Manuscripta Math.* **87** (1995), 501–510.

[2] W.R. Alford, A. Granville and C. Pomerance, 'There are infinitely many Carmichael numbers', *Ann. of Math.* **139** (1994), 703–722.

[3] W.D. Banks and C. Pomerance 'On Carmichael numbers in arithmetic progressions', *J. Aust. Math. Soc.* **88** (2010), 313–321.

[4] A.F. Beardon, 'Sums of powers of integers', *Amer. Math. Monthly* **103** (1996), 201–213.

[5] E. Bedocchi, 'Nota ad una congettura sui numeri primi', *Riv. Mat. Univ. Parma* **11** (1985), 229–236.

[6] J. Bernoulli, *Ars Conjectandi*, Basel, 1713.

[7] D. Borwein, J.M. Borwein, P.B. Borwein and R. Girgensohn, 'Giuga's conjecture on primality', *Amer. Math. Monthly* **103** (1996), 40–50.

[8] J.M. Borwein and E. Wong, '*A survey of results relating to Giuga's conjecture on primality*', Proceedings of the 25th Anniversary Conference of the Centre de Récherches Mathématiques, CECM Preprint Series, 95-035:1–23, 1995; available at `http://discerver.carma.newcastle.edu.au/101/`.

[9] C.B. Boyer, 'Pascal's formula for the sums of powers of the integers', *Scripta Math.* **9** (1943), 237–244.

[10] L. Carlitz, 'The Staudt-Clausen theorem', *Math. Mag.* **34** (1960/1961), 131–146.

[11] S.D. Chowla, 'A new proof of von Staudt's theorem', *J. Indian Math. Soc.* **16** (1926), 145–146.

[12] T. Clausen, 'Lehrsatz aus einer Abhandlung über die Bernoullischen Zahlen', *Astron. Nach.* **17** (1840), 351–352.

[13] P. Damianou and P. Schumer, 'A theorem involving the denominators of Bernoulli numbers', *Math. Mag.* **76** (2003), 219–224.

[14] A.W.F. Edwards, 'A quick route to sums of powers', *Amer. Math. Monthly* **93** (1986), 451–455.

[15] Y. Gallot, P. Moree and W. Zudilin, 'The Erdős-Moser equation $1^k + 2^k + \cdots + (m-2)^k + (m-1)^k = m^k$ revisted using continued fractions', *Math. Comp.* **80** (2011), 1221–1237; preprint `arXiv:0907.1356v1 [math.NT]`, 2009.

[16] G. Giuga, 'Su una presumibile proprietà caratteristica dei numeri primi', *Ist. Lombardo Sci. Lett. Rend. A* **83** (1950), 511–528.

[17] R.L. Graham, D.E. Knuth and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, Addison-Wesley Publishing Company, 1989.

[18] J.M. Grau and A.M. Oller-Marcén, 'Generalizing Giuga's conjecture', preprint `arXiv:1103.3483v1 [math.NT]`, 2011.

[19] J.M. Grau, F. Luca and A.M. Oller-Marcén, 'On a variant of Giuga numbers', *Acta Math. Sin.* (*Engl. Ser.*) **28**, No. 4 (2012), 653–660.

[20] G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, Clarendon Press, Oxford, 1979.

[21] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, Springer-Verlag, New York, 1982.

[22] B.C. Kellner, 'Über irreguläre Paare höherer Ordnungen', `http://www.bernoulli.org/~bk/irrpairord.pdf`, 2002.

[23] B.C. Kellner, 'The equivalence of Giuga's and Agoh's conjectures', preprint `arXiv:math/0409259v1` [math.NT], 2004.

[24] D.E. Knuth, 'Johan Faulhaber and sums of powers', *Math. Comp.* **61** (1993), 277–294.

[25] T. Lengyel, 'On divisibility of some power sums', *Integers* **7** (2007), A# 41, 1–6.

[26] F. Luca, C. Pomerance and I. Shparlinski, 'On Giuga numbers', *Int. J. Mod. Math.* **4** (2009), 13–18.

[27] K. MacMillan and J. Sondow, 'Proofs of power sum and binomial coefficient congruences via Pascal's identity', *Amer. Math. Monthly* **118** (2011), No. 6, 549–552; available at `arXiv:1011.0076v1 [math.NT]`, 2010.

[28] K. MacMillan and J. Sondow, 'Divisibility of power sums and the generalized Erdős-Moser equation', available at `arXiv:1010.2275v4[math.NT]`, 2011.

[29] R.J. McIntosh and E.L. Roettger, 'A search for Fibonacci-Wieferich and Wolstenholme primes', *Math. Comp.* **76** (2007) 2087–2094.

[30] R. Meštrović, 'On the mod p^7 determination of $\binom{2p-1}{p-1}$', accepted for publication in *Rocky Mount. J. Math.*; preprint `arXiv:1108.1174v1 [math.NT]`, 2011.

[31] R. Meštrović, 'Some identities in commutative rings with unity and their applications', submitted.

[32] P. Moree, 'On a theorem of Carlitz-von Staudt', *C. R. Math. Rep. Acad. Sci. Canada* **16** (1994), 166–170.

[33] P. Moree, 'Diophantine equations of Erdős-Moser type', *Bull. Aust. Math. Soc.* **53** (1996), 281–292.

[34] P. Moree, 'A top hat for Moser's four mathemagical rabbits', *Amer. Math. Monthly* **118** (2011), 364–370.

[35] P. Moree, H.J.J. te Riele and J. Urbanowicz, 'Divisibility properties of integers x, k satisfying $1^k + \cdots + (x-1)^k = x^k$', *Math. Comp.* **63** (1994), 799–815.
[36] L. Moser, 'On the Diophantine equation $1^n + 2^n + \cdots + (m-1)^n = m^n$', *Scripta Math.* **19** (1953), 84–88.
[37] H.J. Schultz, 'The sums of the kth powers of the first n integers', *Amer. Math. Monthly* **87** (1980), 478–481.
[38] N.J. Sloane, Sequences A064538 and A079618 in *The On-Line Encyclopedia of Integer Sequences.*
[39] J. Sondow and K. MacMillan, 'Reducing the Erdős-Moser equation $1^n + 2^n + \cdots + k^n = (k+1)^n$ modulo k and k^2', *Integers* **11** (2011), # A34, pages 8.
[40] K.G.C. von Staudt, 'Beweis eines Lehrsatzes die Bernoulli'schen Zahlen betreffend', *J. Reine Angew. Math.* **21** (1840), 372–374.
[41] Z.-H. Sun, 'Congruences for Bernoulli numbers and Bernoulli polynomials', *Discrete Math.* **163** (1997), 153–163.
[42] Z.-H. Sun, 'Congruences concerning Bernoulli numbers and Bernoulli polynomials', *Discrete Appl. Math.* **105** (2000), 193–223.
[43] Z.-H. Sun, 'Congruences involving Bernoulli polynomials', *Discrete Math.* **308** (2008), 71–112.
[44] Z.-H. Sun, 'Congruences involving Bernoulli and Euler numbers', *J. Number Theory* **128** (2008), 280–312.
[45] Z.-W. Sun, 'General congruences for Bernoulli polynomials', *Discrete Math.* **262** (2003), 253–276.
[46] R. Vakil, *A mathematical Mosaic*: *Patterns and Problem Solving.* Brendan Kelly Pub., Burlington, Ont., 1996.
[47] E. Wong, *Computations on Normal Families of Primes*, MSc Thesis, Simon Fraser University, 1997; available at `http://discerver.carma.newcastle.edu.au/view/year/1997.html`.

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